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EXTENDED ABSTRACT

On the determination of friction velocity and Monin-Obukhov length, based on sensible heat flux or vertical temperature difference.

Leif Persson, FOI CBRN Defense and Security, Umeå, Sweden

leiper@foi.se

Fredrik Schönfeldt, FOI CBRN Defense and Security, Umeå, Sweden

Introduction.

A common problem in dispersion modeling is to determine the friction velocity u_* and the Monin-Obukhov length L by solving the Monin-Obukhov similarity equations

$$L = -\frac{u_*^3}{\kappa\beta H} \qquad (1a) \qquad\qquad u_* = \frac{U\kappa}{\log(z_m^{-1}z) - \psi_m(L^{-1}z, L^{-1}z_m)} \qquad (2a)$$

$$H = -\frac{\Delta\Theta u_* \kappa}{\log(z_h^{-1}z) - \psi_h(L^{-1}z, L^{-1}z_h)} \qquad (3a)$$

where U is the wind speed at height z above ground; z_m, z_h are roughness lengths for momentum and heat; ψ_m, ψ_h are Panofsky's integral similarity functions; $\Delta\Theta = \Theta(z) - \Theta(z_h)$; $H = \langle w\theta \rangle$ is the surface sensible heat flux[1]; w, θ are turbulent fluctuations of vertical velocity and temperature; $\beta = \Theta_0^{-1} g$ is a buoyancy parameter; g is gravitational acceleration; Θ_0 is a reference temperature for the surface layer; κ is von Karman's constant. We consider two cases:

i) given H solve (1a), (2a) for (L, u_*)

ii) given $\Delta\Theta$ solve (1a), (2a), (3a) for (L, u_*, H).

In case i), stable stratification ($L > 0$, $H < 0$, $\Delta\Theta > 0$) there are distinct solutions and a minimum feasible H . This was first noted in (Taylor, 1971) and further elaborated in e.g., (Malhi, 1995), (van de Viel et al, 2007) and (Basu et al, 2008). See Figure 1 below for an illustration. The iterative method in (Basu et al, 2008) always captures the less stable solution; to capture the more stable solution we need other methods. In case ii), stable stratification there is a unique solution[2] and a maximum feasible $\Delta\Theta$. Our main contribution in this work is to construct these analytical solutions, extending the results of (Basu et al, 2008) to the case $z_m \neq z_h$ and $\psi_m \neq \psi_h$. Precise statements are provided in Proposition 1 and Proposition 2 below. To this end, we review the derivation of the Monin-Obukhov similarity theory, and provide a modified, nondimensional formulation

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[1] We use the terminology in (Basu et al, 2008) here.

[2] We will see that mathematically, there might be two distinct solutions for some sufficiently large values of $\Delta\Theta$, but these solutions are likely not physically relevant.

of (1a)-(3a), using modified integral similarity functions. Solving for (L, u_*) given (U, H) amounts to find intersection points between isocurves for U and H as functions of (L, u_*). The multiple solutions phenomenon is illustrated in Figure 1.

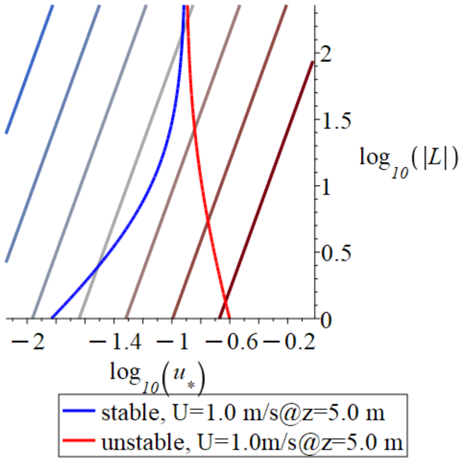

Figure 1. Typical isocurves for H (straight lines), $U(z)$ for stable stratification (blue curve), and $U(z)$ for unstable stratification (red curve) as functions of $\log_{10}(u_*), \log_{10}(|L|)$. Gradients are directed to the right, and if U is increased the U isocurves are translated to the right. U and H isocurves intersect twice (for small $|H|$) or not at all (for large $|H|$) in the stable case, and exactly once in the unstable case. Both U curves have the vertical asymptote $\log_{10} u_{*N}$ where u_{*N} is the friction velocity in the neutral limit.

Derivation of Monin-Obukhov similarity equations.

The Monin-Obukhov equations are derived by dimensional analysis (Barenblatt, 1997) assuming that the vertical average wind speed and temperature gradients depend on $u_*, \beta, \nu, H, z, \Lambda, \chi$. Here Λ is the height of the surface layer where u_* and H are assumed to be approximately constant; ν is the kinematical viscosity of air; χ is the thermal molecular diffusivity of air. Selecting the variables u_*, β, ν as fundamental units we get the length, time and temperature scales $\nu u_*^{-1}, \nu u_*^{-2}, u_*^3 \beta^{-1} \nu^{-1}$. The remaining variables H, z, Λ, χ are scaled to nondimensional form

$$\Pi_H = \beta \nu u_*^{-4} H, \quad \Pi_z = \mathrm{Re}_\ell = u_* \nu^{-1} z, \quad \Pi_\Lambda = \mathrm{Re}_* = u_* \nu^{-1} \Lambda, \quad \Pi_\chi = \mathrm{Pr} = \nu^{-1} \chi,$$

where $\mathrm{Re}_\ell, \mathrm{Re}_*, \mathrm{Pr}$ are known as the local Reynolds number, the global Reynolds number, and the Prandtl number, respectively. Replace Π_H by the ν-free nondimensional variable[3] $\kappa \Pi_H \Pi_z = \kappa \beta u_*^{-3} z = \zeta$. Now the nondimensional gradients $\Pi_{\partial_z U} = \kappa z u_*^{-1} \partial_z U$, $\Pi_{\partial_z \Theta} = -\kappa z u_* H^{-1} \partial_z \Theta$ must be functions of $\zeta, \mathrm{Re}_\ell, \mathrm{Re}_*, \mathrm{Pr}$ by the Buckingham Pi theorem. Assuming complete similarity in the limit of infinite Reynolds numbers we obtain

$$\frac{\kappa z}{u_*} \frac{\partial U}{\partial z} = \phi_m(\zeta) \qquad (4) \qquad\qquad -\frac{\kappa u_* z}{H} \frac{\partial \Theta}{\partial z} = \phi_h(\zeta) \qquad (5)$$

Here ϕ_m, ϕ_h are the Monin-Obukhov similarity functions, cf. (Foken, 2006).

[3] The factor κ is added by convention.

Derivation of profile functions

To derive velocity and temperature profiles Panofsky defined the following integral similarity functions, cf. (Kramm et al, 2013)

$$\psi(\zeta_2,\zeta_1)=\int_{\zeta_1}^{\zeta_2}\zeta^{-1}(1-\phi(\zeta))d\zeta, \tag{6}$$

which yields the velocity and temperature profiles in disguise (1), (3) by integration

$$\int_{\zeta_1}^{\zeta_2}\zeta^{-1}\phi(\zeta)d\zeta=\ln\left(\zeta_1^{-1}\zeta_2\right)-\psi(\zeta_2,\zeta_1). \tag{7}$$

In the context of this paper, it is more natural to use modified integral similarity functions defined by

$$\hat{\psi}(\zeta_2,\zeta_1)=1-(\ln(\zeta_1^{-1}\zeta_2))^{-1}\psi(\zeta_2,\zeta_1)=(\ln(\zeta_1^{-1}\zeta_2))^{-1}\int_{\zeta_1}^{\zeta_2}\zeta^{-1}\phi(\zeta)d\zeta \tag{8}$$

which yields a multiplicative rather than additive correction to the logarithmic term

$$\int_{\zeta_1}^{\zeta_2}\zeta^{-1}\phi(\zeta)d\zeta=\hat{\psi}\left(\zeta_2,\zeta_1\right)\ln\left(\zeta_1^{-1}\zeta_2\right).$$

Nondimensional formulation and general solution

Using the modified integral similarity functions, we rewrite the system (1a)-(3a) on the equivalent nondimensional form[4]

$$\zeta=-\frac{\hat{H}}{\hat{u}_*^3} \quad (1b) \qquad \hat{u}_*=\frac{1}{\hat{\psi}_m(\zeta,\xi_m^{-1}\zeta)} \quad (2b) \qquad \hat{H}=-\frac{\Delta\hat{\Theta}\hat{u}_*}{\hat{\psi}_h(\zeta,\xi_h^{-1}\zeta)} \quad (3b)$$

where the nondimensional variables are defined by

$$\zeta=\frac{z}{L} \quad (9) \qquad \xi_m=\frac{z}{z_m} \quad (10) \qquad \xi_h=\frac{z}{z_h} \quad (11)$$

$$\hat{u}_*=\frac{\ln\xi_m}{\kappa}\frac{u_*}{U} \quad (12) \qquad \hat{H}=\frac{\beta z(\ln\xi_m)^3}{\kappa^2}\frac{H}{U^3} \quad (13) \qquad \Delta\hat{\Theta}=\frac{\beta z\ln\xi_m}{\kappa\ln\xi_h}\frac{\Delta\Theta}{U^2} \quad (14)$$

Solutions are derived as follows.

Case i): Eliminating $\hat{u}_*$ in (1b) we see that $(\zeta,\hat{u}_*)$ satisfies (1b), (2b) if and only if $\hat{u}_*$ satisfies (2b) and

$$\hat{\psi}_m(\zeta,\xi_m^{-1}\zeta)=\left(-\hat{H}^{-1}\zeta\right)^{1/3} \quad (1bi)$$

Case ii): Eliminating $\hat{u}_*,\hat{H}$ in (1b) we see that $(\zeta,\hat{u}_*,\hat{H})$ satisfies (1b), (2b), (3b) if and only if $\hat{u}_*$ satisfies (2b) and

$$\frac{\hat{\psi}_m^2(\zeta,\xi_m^{-1}\zeta)}{\hat{\psi}_h(\zeta,\xi_h^{-1}\zeta)}=\frac{\zeta}{\Delta\hat{\Theta}} \quad (1bii) \qquad \hat{H}=-\frac{\zeta}{\hat{\psi}_m^3(\zeta,\xi_m^{-1}\zeta)} \quad (3bii)$$

[4] Note that $\hat{H}$ as defined in (Basu et al, 2008) for stable stratification differs from our definition by a factor $-\gamma_m/\ln(z/z_m)$.

Analytical solutions for stable stratification.

For moderately stable stratifications ($\zeta > 0$, $\hat{H} < 0$, $\Delta\hat{\Theta} > 0$), the similarity functions ϕ_m, ϕ_h are linear (Kramm et al, 2013): $\phi(\zeta) = \alpha + \gamma\zeta$, which yields

$$\hat{\psi}(\zeta, \xi^{-1}\zeta) = \alpha + \hat{\gamma}\zeta, \qquad (16)$$

where

$$\hat{\gamma} = \gamma / f(\xi), \quad f(\xi) = (1-\xi^{-1})^{-1} \ln \xi. \qquad (17)$$

A solution is said to be feasible if $0 < \zeta \le \zeta_e \approx 1$. Our main results are formulated in the following two propositions.

Proposition 1 (case i) Assume that $\phi_m(\zeta) = 1 + \gamma_m\zeta$. Let $x_c = 2^{-1/3}$, $p_c = 3 \times 2^{-2/3}$, $\zeta_c = \hat{\gamma}_m^{-1} x_c^3$, $\hat{H}_c = -\hat{\gamma}_m^{-1} p_c^{-3}$, $x_e = (\hat{\gamma}_m \zeta_e)^{1/3}$, $p_e = x_e^{-1}(x_e^3 + 1)$, $\hat{H}_e = -\hat{\gamma}_m^{-1} p_e^{-3}$, $p = (-\hat{\gamma}_m \hat{H})^{-1/3}$, $\vartheta = (1/3)\arccos\left(-(3/p)^{3/2}/2\right)$,

$$x_k = 2\sqrt{p/3}\cos\left(\vartheta - 2\pi k/3\right), \quad k = 0,1 \qquad (18)$$

and $\zeta_k = \hat{\gamma}_m^{-1} x_k^3$. Then, in case ζ_c is infeasible, (1bi) has no feasible solutions if $\hat{H} < \hat{H}_e$ and one feasible solution $\zeta = \zeta_1$ if $\hat{H} \ge \hat{H}_e$. Moreover, in case ζ_c is feasible, (1bi) has no feasible solutions if $\hat{H} < \hat{H}_c$, one feasible solution $\zeta = \zeta_1$ if $\hat{H} = \hat{H}_c$ or $\hat{H} > \hat{H}_e$, and two feasible solutions $\zeta \in \{\zeta_0, \zeta_1\}$ if $\hat{H}_c < \hat{H} \le \hat{H}_e$. In the latter case, the two feasible solutions satisfy $\zeta_1 \le \zeta_c \le \zeta_0$ and the corresponding solutions $\hat{u}_{*0}, \hat{u}_{*1}$ given by (2b) satisfy $\hat{u}_{*0} \le \hat{u}_{*c} \le \hat{u}_{*1}$ where $\hat{u}_{*c} = 2/3$.

Proof. Note first that p is strictly increasing with $\hat{H}$. Substituting $\zeta = \hat{\gamma}_m^{-1} x^3$ yields the *depressed cubic equation* $x^3 + 1 = px$ having three distinct real solutions if and only if the discriminant $27 - 4p^3 < 0$, i.e., $p > p_c$. In this case *Viètes formula* yields the positive solutions (18), which are feasible if and only if $x_k \le x_e$. In case $p = p_c$ there is a positive double root $x = x_c$ and in case $p < p_c$ there is only one real root, which is negative. The bounds on p are readily translated into the claimed bounds on $\hat{H}$, which concludes the proof.

We remark that ζ_c is feasible if and only if $x_c \ge x_e$, or equivalently, $\hat{\gamma}_m \ge x_c^3 \zeta_e^{-1}$, or equivalently, $\xi_m \le f^{-1}\left(x_c^{-3}\zeta_e\gamma_m\right)$. The inverse function can be expressed in terms of branches of the *Lambert W-function*:

$$f^{-1}(y) = 1_{\{0\le y\le 1\}} \exp\left(W_{-1}(-y\exp(-y)) + y\right) + 1_{\{1\le y\}} \exp\left(W_0(-y\exp(-y)) + y\right). \qquad (19)$$

Proposition 2 (case ii) Assume that $\phi_m(\zeta)=1+\gamma_m\zeta$ and $\phi_h(\zeta)=\alpha_h+\gamma_h\zeta$. Let $\eta=2\left(1-\alpha_h^{-1}\hat{\gamma}_m^{-1}\hat{\gamma}_h\right)$, $p_c=\max(0,\eta)$, $x_c=p_c-1, \zeta_c=\left(\hat{\gamma}_m x_c\right)^{-1}$, $\Delta\hat{\Theta}_c=\left(2\hat{\gamma}_m p_c\right)^{-1}\alpha_h$, $x_e=\left(\hat{\gamma}_m\zeta_e\right)^{-1}$, $p_e=\left(2(x_e+1)-\eta\right)^{-1}\left(x_e+1\right)^2$, $\Delta\hat{\Theta}_e=\left(2\hat{\gamma}_m p_e\right)^{-1}\alpha_h$, $p=\left(2\hat{\gamma}_m\Delta\hat{\Theta}\right)^{-1}\alpha_h$,

$$x_k=p-1-(-1)^k\sqrt{p(p-\eta)},\quad k=0,1 \tag{20}$$

and $\zeta_k=\left(\hat{\gamma}_m x_k\right)^{-1}$. Then, in case ζ_c is infeasible, (1bii) has no feasible solutions if $\Delta\hat{\Theta}>\Delta\hat{\Theta}_e$ and one feasible solution $\zeta=\zeta_1$ if $\Delta\hat{\Theta}\le\Delta\hat{\Theta}_e$. Moreover, in case ζ_c is feasible, (1bii) has no feasible solutions if $\Delta\hat{\Theta}>\Delta\hat{\Theta}_c$, one feasible solution $\zeta=\zeta_1$ if $\Delta\hat{\Theta}=\Delta\hat{\Theta}_c$ or $\Delta\hat{\Theta}<\Delta\hat{\Theta}_e$, and two feasible solutions $\zeta\in\{\zeta_0,\zeta_1\}$ if $\Delta\hat{\Theta}_e\le\Delta\hat{\Theta}<\Delta\hat{\Theta}_c$.

Proof. Note first that p is strictly decreasing with $\Delta\hat{\Theta}$. Substituting $\zeta=\left(\hat{\gamma}_m x\right)^{-1}$, multiplying both sides with the denominator and simplifying we get the quadratic equation $\left(x+1\right)^2-2p\left(x+1\right)+p\eta=0$ with solutions (20), which defines a two-valued function $x(p)$ for $p\ge p_c$ with an increasing, concave upper branch $x_1(p)$ with limit $+\infty$ as $p\to+\infty$, and a decreasing, convex lower branch $x_0(p)$ with limit $\eta/2-1=-\alpha_h^{-1}\hat{\gamma}_m^{-1}\hat{\gamma}_h<0$ as $p\to+\infty$, and $x_0(p_c)=x_1(p_c)=x_c$. Solutions are feasible if $x(p)\ge x_e>0$. If x_c is infeasible, all points on the upper branch with $p\ge p_e$ are feasible and all other points are infeasible. If x_c is feasible, all points on the upper branch with $p\ge p_c$ are feasible, as well as all points on the lower branch with $p_c\le p\le p_e$, and all other points are infeasible. The bounds on p are readily translated into the claimed bounds on $\Delta\hat{\Theta}$, which concludes the proof.

Finally, we remark that ζ_c is feasible if and only if $x_c\ge x_e$, or equivalently, $\eta\ge 1+x_e$, or equivalently, $\hat{\gamma}_m\ge 2\alpha_h^{-1}\hat{\gamma}_h+\zeta_e^{-1}$, or equivalently, with f^{-1} given by (19): $\xi_m\le f^{-1}\left(\left(2\alpha_h^{-1}f\left(\xi_h\right)^{-1}\gamma_h+\zeta_e^{-1}\right)^{-1}\gamma_m\right)$. This requires $\xi_m\ll\xi_h$ by several orders of magnitude for common values of $\gamma_m,\gamma_h,\alpha_h$, and is thus likely not physically relevant.

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