REPRESENTING THE DISPERSION OF EMISSIONS FROM AIRCRAFT ON RUNWAYS

A. Graham, M. Bennett and S. Christie

Centre for Air Transport and the Environment (CATE), Manchester Metropolitan University, Manchester, United Kingdom

Abstract: Aircraft in their takeoff ground run constitute an unavoidable strong source of emissions subject to a highly variable motion. The intermittent nature of release makes it difficult to measure the emissions and establish their impact on mean concentrations in the vicinity of airports. Practically, it is also difficult to perform experimental studies near taxiways and runways. Remote observations have nevertheless recently been obtained by a rapidly-swept UV Lidar, and analysis of these has necessitated and informed a parallel modelling effort. Aircraft exhausts disperse in a complex manner, as they are subject not only to transport processes of the ambient atmosphere, but also to those associated with the aircraft itself (with diffusion in ambient turbulence to be expected once turbulence resulting from the aircraft falls to ambient levels). They have a downstream Lagrangian momentum associated with the engine thrust, and steadily acquire a vertical momentum as a result of their buoyancy. Exhaust streams merge and interact strongly with the ground to form a common emission plume within about a wingspan downstream of engines' exits. Before the aircraft reaches a threshold speed, is rotated upward and lifts off, the downstream (thrust) forcing and upstream source acceleration are approximately constant, and a first-order nonlinear partial differential equation may be expressed capturing the turbulent diffusion of the plume in the reference frame of the source. The downstream forcing exceeds the buoyant forcing, so the plume remains in contact with the ground, but is heightened and narrowed by buoyant rise. During rotation and liftoff, the net downstream forcing declines as a significant airframe drag arises, and the source acceleration plummets. More importantly, lift on the airframe and the associated shed circulation cause exhausts to move downward and, in proximity to ground, outward, so their dynamics decouple from those of exhausts released earlier, with re-coupling unlikely before the aircraft has turned in its flight path. Once rotation is initiated, this argues for a simplifying (and partly analytic) treatment of the exhaust plume generated earlier, such that the turbulent diffusion of a given elemental plume segment is taken to match that of an infinitely long flow tube (with the latter ascribed the same buoyancy density and downstream Lagrangian momentum density – or mass flux – throughout).

1. INTRODUCTION

To satisfy statutory planning procedures and national and international directives, the impact of a major airport development on air quality as well as noise levels must be assessed. One of the most difficult issues facing construction of a third runway at London Heathrow, for example, is meeting statutory EU limits on NO_2.

Aircraft in ground run and flight constitute an unavoidable, strong source of emissions subject to a highly variable motion. The intermittent nature of release makes it difficult to measure the emissions and establish their impact on mean concentrations in the vicinity of airports. Prediction of concentrations is further hindered by the limited scope of standard emission tests, which do not address the form or composition of particulates or the speciation of NO_x at the engine exit. There are also important uncertainties in how emissions disperse before they may be considered subsumed within an ambient contaminant field. Airports are often sited at the urban fringe, where the aerodynamic roughness length varies markedly with wind direction. Exhausts are not subject solely to ambient transport processes, moreover, but also to those resulting from the aircraft itself. They have a Lagrangian (excess) momentum in the downstream direction (opposite to aircraft heading), as a result of the engine thrust, and acquire a vertical momentum as a result of their buoyancy. They may reside within an airframe wake, moving downward or, in the vicinity of ground, outward, as a result of the lift generated and circulation shed.

Emissions released during the takeoff ground run, when engines work hardest, and which may carry a high air-quality penalty, are addressed here. Owing to the number of potential variables, simple empirical fits to suitable field data cannot realistically be pursued. There is thus need for predictions from a model, accurate to $O(1)$ in situations where air quality may be an issue, for validation to such data. During the ground run, the upstream acceleration of the aircraft means the mean flow in the plume is unsteady, and unlikely to yield to closed-form analytic solution.

2. THEORY

Takeoff ground run

Exhaust streams from aircraft engines may be taken to merge as they interact with one another and the ground so as to yield a common plume within about a wingspan downstream of engines' exits (this likely enhanced through a Coanda effect). In uniform and still ambient conditions, the plume's characteristic cross-sectional area, A, and downstream Lagrangian velocity, Δu, and the age of its emissions, t, will become asymptotically independent of source scales and mass flux and any excess pressure at the source, but retain sensitivity to the momentum forcing. The source force, F, may be taken as the combined static thrust from engines. (Minor changes in the thrust and aircraft acceleration may be ignored prior to the time that the aircraft is rotated upward and lifts off.) In an elemental segment of the plume of length, dx, a distance, x, downstream of the effective source (at the engine exit plane), the excess downstream momentum is equal to the following, $\rho A \Delta u \mathrm{d}x$,

where ρ is the ambient air density (1.2 kg m^{-3} under ISA sea-level conditions, with departures from this within the plume being dynamically insignificant here). With the characteristic spread of emission ages over dx being given by $\mathrm{d}x \cdot \partial t/\partial x$, the excess downstream momentum may alternatively be expressed as the following

$$k_F F \mathrm{d}x\, \partial t/\partial x\,,$$

where k_F is a fixed fraction accounting for drag losses upstream, most of the momentum being lost near the source. (Momentum from a steady fixed source at a wall passes through the jet cross-section at a decreasing rate with increasing distance from the source, the decrease per unit x scaling with F/x.) Equating terms and rearranging, it can be seen that

$$k_F (F/\rho)\partial t/\partial x = A\Delta u \tag{1}$$

A relation between Δu and t follows from the Chain Rule,

$$\mathrm{d}t/\mathrm{d}T = 1 = \partial t/\partial T + (aT + \Delta u)\,\partial t/\partial x\,, \tag{2}$$

where a is the aircraft acceleration and T the time since the ground run was begun (from rest, it may be assumed to $O(1)$). A corresponding relation between A and t may be obtained via an entrainment approximation, taking A to grow at a rate scaling with a characteristic local eddy viscosity (or diffusivity). Entrainment is governed (indirectly) by the most energetic eddies, of size equal to the maximum scale of isotropy in the turbulence, $A^{1/2}$. The relative importance of the momentum and buoyancy of exhausts in determining the strength of the eddies follows from a time scale, τ, equal to the ratio of F to the buoyant force imparted to exhausts per unit time at the source. This is calculated in the Appendix to be a factor, 2.3, times c/g, where c is the speed of sound under ISA sea-level conditions and g the acceleration due to gravity. Thus, τ=80 s, which, given that the ground run lasts for some 30-45 s, must be somewhat larger than T. The downstream forcing will consequently dominate in the ground run, with the plume likely to remain in contact with the ground, and with a characteristic turbulent velocity thus being ascribable to the eddies that scales with Δu alone. It follows that

$$\mathrm{d}A^{1/2}/\mathrm{d}T = k_\varepsilon \Delta u\,, \tag{3}$$

where k_ε is a constant coefficient. Speed Δu is the rate of increase of the downstream Lagrangian displacement, x_L,

$$x_L = x - \left[a(T-t)t + at^2/2\right] = x - a(T - t/2)t\,, \tag{4}$$

and so, with A following from Equation (3) as a fraction, $k_\varepsilon^{\,2}$, times $x_L^{\,2}$, it can be seen that

$$A = k_\varepsilon^{\,2}[x - a(T-t/2)t]^2 \tag{5}$$

Substituting for Δu and A in Equation (1) using Equations (2) and (5), respectively, a first-order p.d.e. in t is obtained,

$$\left(k_\varepsilon^{\,2}/k_F\right)(1 - \partial t/\partial T - aT\,\partial t/\partial x)[x - a(T-t/2)t]^2/(F/\rho) - (\partial t/\partial x)^2 = 0\,, \tag{6}$$

with t and its partial derivatives tending to zero as x tends to zero. Once t and its derivatives are known, Δu and A follow trivially from Equations (2) and (5).

A mass balance may analogously be constructed for a conservative tracer emitted by the source at a steady rate, Q_t, and arising in the plume at a characteristic concentration, χ. The tracer mass in an elemental segment of the plume is equal to $\chi A\mathrm{d}x$; or, alternatively, to

$$Q_t \mathrm{d}x \cdot \partial t/\partial x\,.$$

Thus,

$$\chi = (Q_t\, \partial t/\partial x)/A\,. \tag{7}$$

Substituting in for $(\partial t/\partial x)/A$ using equation (1), it can be seen that

$$\chi/\rho = (1/k_F)Q_t \Delta u/F\,. \tag{8}$$

Substituting for Δu from Equation (2),

$$\chi/\rho = (1/k_F)[1 - \partial t/\partial T - aT\,\partial t/\partial x](Q_t/F)/(\partial t/\partial x)\,. \tag{9}$$

Concentration χ consequently follows once t and its derivatives are known.

A vertical momentum balance may also be constructed, in light of emission buoyancy. The downstream flow forces an inflow from an irrotational flow above it, replacing fluid previously entrained and swept away, with a correspondingly reduced pressure existing at the plume's upper surface. The difference in modified (i.e. static-offset) pressure between the plume's lower (boundary-flush) and upper faces counterbalances the influx of vertical momentum, so that, in the absence of emission buoyancy, the mean (vertically-integrated) flow in the plume is essentially parallel to the boundary. The buoyancy leaves A, Δu and velocities and modified pressures at plume boundaries unchanged to $O(1)$ (as $t \ll \tau$), but will tend to force an upward deflection of mean streamlines, through the action of a characteristic rise speed, v. Now the buoyant momentum in an elemental segment of the plume is equal to $\rho A v \mathrm{d}x$; or, alternatively, to

$$(F/\tau)\,t\,\mathrm{d}x \cdot \partial t/\partial x\,.$$

Thus,

$$v = (F/\rho)(t/\tau)(\partial t/\partial x)/A\,. \tag{10}$$

Substituting in for $(\partial t/\partial x)/A$ using equation (1), it can be seen that

$$v/\Delta u = (1/k_F)\,t/\tau\,. \tag{11}$$

When t is much smaller than τ, and buoyancy is dynamically inactive, the characteristic plume width, Δy, and mean height, z, likely scale with x_L. The scaling may be that of a wall jet, with Δy being up to an order of magnitude greater than z. When t is within an order of τ, however, z will come to approximate a height of buoyant rise, so that

$$v = \mathrm{d}z/\mathrm{d}T = \partial z/\partial T + (aT + \Delta u)\partial z/\partial x \; . \tag{12}$$

Substituting this into Equation (11), rearranging and substituting for Δu from Equation (2) leads to a first-order p.d.e.,

$$(\partial t/\partial x)\cdot(\partial z/\partial T) + (1 - \partial t/\partial T)\partial z/\partial x - (1/k_F)(1 - \partial t/\partial T - aT\,\partial t/\partial x)t/\tau = 0\,, \tag{13}$$

with z and its derivatives tending to zero as x tends to zero. Height z may thus be found on solution of Equation (6); Δy scales with A/z.

Cessation of source forcing

Exhausts will cease to evolve in the manner described previously when the aircraft is rotated upward and becomes airborne. At liftoff, the net downstream forcing is half that at the start of the ground run, and the upstream acceleration of the aircraft has fallen by an order of magnitude. More importantly, lift on the airframe and the associated shed circulation cause exhausts to move downward and, through the proximity to ground, outward, so their dynamics decouple from those of exhausts released earlier, with re-coupling unlikely before the aircraft has turned in its flight path. Once rotation is initiated, this argues for a simplifying treatment of the exhaust plume generated earlier, such that the turbulent diffusion of a given elemental plume segment is taken to match that of an infinitely long flow tube (the latter being ascribed the same buoyancy density and downstream Lagrangian momentum density – or mass flux – throughout). The flow will in reality cease to be essentially tubular when cross-sectional scales or Lagrangian displacements approach the length of the plume at the onset of rotation. It will, however, be hard by then to distinguish concentrations from ambient, and diffusion in ambient turbulence will in any case generally be in effect (as addressed shortly).

In the proposed idealisation of the flow, from the initiation of upward rotation of the aircraft, exhausts released earlier are taken to reside within a flow tube, of axial Lagrangian momentum density, I_L (the axial momentum per unit tube length), and buoyancy density, J_L (the force from emission buoyancy per unit tube length). These densities are thereafter invariant with time, T (ignoring drag of the plume at the ground as previously). The characteristic cross-sectional area of the tube, A, and axial Lagrangian speed, Δu, then satisfy

$$I_L = \rho A \Delta u = \rho A_R \Delta u_R\,, \tag{14}$$

where A_R, Δu_R represent values at the onset of rotation. The constancy of axial Lagrangian mass flux means there can be no cross-flow into the tube (and thus no associated Coanda effect). Cross-flows outside the tube arise from the buoyant forcing alone, and are such that fluid particles escaping entrainment as the streamtube rises past them tend to return to their original height but move inward, toward the rise axis, to replace the entrained fluid. The net vertical momentum per unit tube length, as integrated over the cross-plane, thus resides wholly within the tube, and satisfies

$$J_L(T - T_R) = \rho(Av - A_R v_R)\,, \tag{15}$$

where v is the characteristic rise speed of the tube, and T_R and v_R represent values at the onset of rotation. (Note that v is now free to dominate over Δu at sufficiently large values of T.) Density J_L satisfies

$$J_L = (F/\tau)\,\partial t/\partial x|_{T_R} = (1/k_F)\,\rho A_R \Delta u_R/\tau\,, \tag{16}$$

$$= (1/k_F)\,I_L/\tau\,, \tag{17}$$

with the second step in equation (16) following on use of Equation (1), and Equation (17) following on use of Equation (14).

To solve for A, Δu and v, a relation in addition to Equations (14) and (15) is required. This may be obtained, as before, via an entrainment approximation, and generalised analogue of Equation (3),

$$\mathrm{d}A^{1/2}/\mathrm{d}T = k_\varepsilon\left(\Delta u^2 + v^2\right)^{1/2} \tag{18}$$

Equations (14), (15) and (18) then have the following solution,

$$(A/A_R)^{3/2} = 1 + (3k_\varepsilon/2)\left[I_L^{\,2}\big/\left(\rho J_L A_R^{3/2}\right)\right]\times \left[\left(1+S^2\right)^{1/2}S - \left(1+s^2\right)^{1/2}s + \log_e\left\{\left[\left(1+S^2\right)^{1/2}+S\right]\big/\left[\left(1+s^2\right)^{1/2}+s\right]\right\}\right], \tag{19}$$

$$\Delta u/\Delta u_R = A_R/A\,, \tag{20}$$

$$v/v_R = J_L(T - T_R)/(\rho A v_R) + A_R/A\,, \tag{21}$$

$$S \equiv J_L(T - T_R)/I_L + s\,, \tag{22}$$

$$s \equiv \rho A_R v_R/I_L\,. \tag{23}$$

The concentration of a conservative tracer, χ, may also be evaluated on solving for A, according to

$$\chi/\chi_R = A_R/A\,, \tag{24}$$

where χ_R is the value at the onset of rotation. The mean height of buoyant rise, z, follows (numerically) as the integral of v with respect to T. The characteristic width of the streamtube, Δy, may be taken to scale with A/z^2. When z compares with Δy, and v with Δu, a separation of the tube from the ground may be expected (in still and neutral ambient conditions). The width and depth (or characteristic vertical spread) of the tube may thereafter be expected to remain comparable. On asymptotically limiting time scales, $T \gg T_R$, τ, it can be seen that A, Δu, v and χ scale with $[(J_L/\rho)T^2]^{2/3}$, $I_L/(\rho J_L^2 T^4)^{1/3}$, $[(J_L/\rho)/T]^{1/3}$ and $\chi_R A_R/[(J_L/\rho)T^2]^{2/3}$, respectively.

Environmental influences

A wind, if present, may effect more than simple advection. It will augment the downstream flow relative to the aircraft (airports generally operate with aircraft heading into the wind to maximise airspeeds and thus lift), with a reduction in Δu following on conservation of momentum. In air-quality issues, the focus tends to be on light to moderate winds, however, when the strong acceleration of the aircraft (some 2-3 m s^{-2}) means aircraft ground speeds will quickly come to dominate. A diffusion in ambient turbulence will also effectively begin once $\mathrm{d}A^{1/2}/\mathrm{d}t$ is on the order of the friction velocity, though again, this may be discounted during the ground run in light to moderate winds. An ambient thermal forcing may similarly be ignored in the run. It may be accommodated thereafter within an expanded treatment, though space precludes this here.

3. VALIDATION

Exhausts from aircraft at airports are generally invisible, so observations on their physical characteristics have recently been obtained remotely at Heathrow and Manchester Airports with a backscatter Lidar, in the near UV (λ=355 nm). Takeoffs are studied with the beam typically being swept rapidly upward from an elevation of 2º or less (at ~10º s^{-1}), with the observation plane intersecting the extrapolated runway centreline some 100-300 m downstream of engines at their initial position. Near-instantaneous vertical sections through the aerosol scattering field are thus obtained, with 4-5 s separating successive sections. Exhaust plumes are evident as regions of enhanced scattering. A computational analysis has been used to remove the background (with geometric spreading and atmospheric extinction being allowed for), and to educe these regions' scales.

A clear separation of plumes from the ground, due to exhaust buoyancy, is rarely evident before the backscatter from exhausts approaches ambient, as exhausts spread and aerosol evaporate. Indeed, the statistics on plume scales obtained generally prove unreliable by the time the aircraft becomes airborne. Comparison with the Theory is thus limited to the aircraft-forced phase of dispersion. A set of 21 takeoffs at Heathrow has been analysed, with Lidar sections obtained at oblique azimuths to the runway. The termination of the exhaust plume at a 'head', wherein exhausts released during power-up for takeoff are concentrated, and the advance of this downstream are observed. The vertical extent of the head has been evaluated, with the analysis discontinued once this ceases to increase monotonically with time. A set of 36 takeoffs at Manchester has also been analysed. Sections there constitute (or approximate to) cross-sections through the plume. The mean height of (excess) backscatter and total (excess) backscatter as integrated over the section have been evaluated, with each takeoff time series being characterised through the mean height from the section of maximum total backscatter.

Predictions for comparison with observation are obtained by solving (6) and (13) numerically, in a dimensionless forward-marching scheme. Values for coefficients k_F and k_ε of 0.7 and 0.3, respectively, are adopted in the comparison with the Manchester results, as following from laboratory study of a turbulent jet from a steady fixed source at a wall over Lagrangian downstream distances 1-2 orders greater than initial cross-sectional scales (Law and Herlina, 2002). Some enhancement of wall drag and entrainment rates might be anticipated in the plume head, as a result of flow rollup, so k_F and k_ε are both assigned a value of 0.4 in the comparison with the Heathrow results. (Any change of coefficients with distance or time is beyond the scope of the Theory.) To match predictions to times since power-up for takeoff, T, of observations, supporting data from the ground radar at Heathrow and footage shot from a terminal building at Manchester have been drawn upon. The footage also yields the downstream distance of the observation plane at the extrapolated runway centreline from engines' exit plane at power-up, x_0. Time T and x_0 enter the simulation as normalised on scales that depend on combined engine thrust (as equilibrated static equivalent), F, and (peak) aircraft acceleration, a. At Heathrow, F is accordingly estimated as a fraction, 0.9, of the top thrust, and a as an invariant 2 m s^{-2} (power plant are known but radar data do not yield a accurately). At Manchester, F and a are estimated from the footage. Data on winds at Manchester have also been used, as wind must be allowed for in predicting plume scales at a fixed location. (Emission drift in the fixed frame is simulated, approximately, as the sum of a jet-forced Lagrangian drift in the absence of wind and a drift in the mean wind alone: see Theory.)

Observation and theory compare as shown in Figure 1. Given that observations from instantaneous sections through plumes are subject to turbulent fluctuation, agreement appears fair. The setup at Heathrow means time series should collapse to universal form (Figure 1a). At the start of the takeoff run, when dimensionless times are on the order 1 or less, the vertical extent of the plume head is predicted to grow through emission buoyancy as if the source were static, as $T^{3/2}$. Later on, effective powers fall. There is some underprediction of observations, but these at the same time suggest a positive y-intercept, so some static fuel burn before engines equilibrate and brakes are released may underlie this. (Underprediction in the limit $T \to 0$ is indeed to be expected, with the plume then approximating a neutral jet, and with near-source effects limiting.)

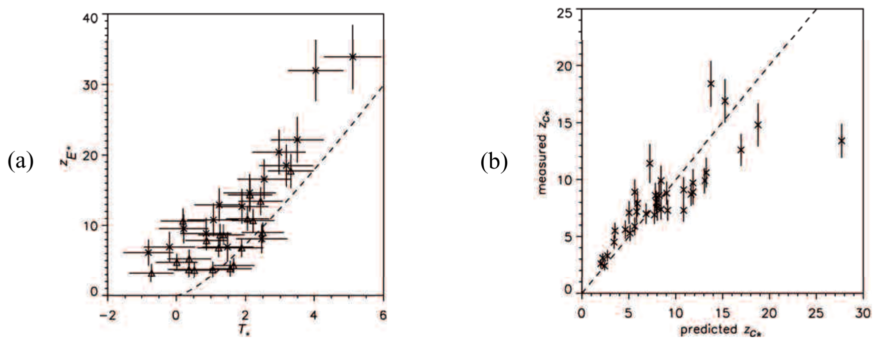

Figure 1. Plot of heights characterising the exhaust plume from aircraft commencing their takeoff ground run, a) extent of plume head from ground, z_E, vs time since start of takeoff, T, b) measured versus predicted mean height, z_c. In (a), observations derive from Lidar sections at Heathrow Airport and are shown as triangles where aircraft are of top thrust 500 kN or more, as crosses for aircraft of lower top thrust. (Two runs commence at abscissa values of less than −1 and appear anomalous, with errors in radar timings more likely responsible than long static fuel burns, so these periods are omitted.) The dashed line shows the predicted dependence, on taking the head to extend to twice its mean height. In (b), observations derive from Lidar sections at Manchester Airport. Sections do not extend wholly to the ground, and predictions have been inflated accordingly. The dashed line corresponds to equivalence. Bars show estimated measurement errors. Asterisks denote dimensionless values, with heights normalised on a scale, $(F/\rho)^{1/2}/(a\tau)$, times on a scale, $[F/(\rho a^4)]^{1/6}$ (see text).

4. CONCLUSIONS

A model has been developed of how emissions released during the takeoff ground run disperse, so that impacts on air quality may be assessed. Turbulent diffusion results from exhaust momentum and buoyancy, and is intrinsically unsteady as a result of the acceleration of the aircraft. A set of universal relations is exploited, so impacts on mean concentrations, averaging over a range of aircraft and conditions, may be easily studied. Data, including from dedicated Lidar trials, corroborate the findings.

Exhausts are found to merge to form a common plume, which their buoyancy then heightens and narrows, but does not cause to separate from ground before the aircraft itself becomes airborne. Separation may instead occur in low winds when the exhaust age is on the order of a scale, τ, equal to a factor, 2.3, times c/g, where c is the speed of sound under ISA sea-level conditions and g the acceleration due to gravity. This works out at 80 s. The plume terminates at a head whose vertical extent comes to grow at a roughly steady rate in the ground run, expressible as a factor, 5, times $[F/(\rho a)]^{1/3}/\tau$, where F is the total thrust from engines, ρ the air density and a the aircraft acceleration. (The run lasts much longer than a scale, $[F/(\rho a^4)]^{1/6}$). This is 1-2 orders larger than would follow from a growth in ambient turbulence in the trials (Webb, 1982; Högström, 1988).

APPENDIX: Buoyancy flux from an aircraft power plant

Propulsive work is done on an aircraft engine by the airflow over an airspace extending a characteristic distance downstream of the exit, x_c, of up to a few exit diameters. Downstream of x_c, the flow is effectively incompressible, and the flux of buoyancy has equilibrated. The pressure at x_c is to $O(1)$ the undisturbed pressure, and so it can be seen that the buoyant force imparted to exhausts in unit time at x_c, B_{Fe}, satisfies

$$B_{Fe} = -Q_{pe}\Delta\rho\, g/\rho = Q_{pe}\Delta\theta\, g/\theta\,, \tag{A.1}$$

where Q_{pe} is the mass flow rate in the plume at x_c; $\Delta\rho$ the characteristic excursion over ambient density, ρ; $\Delta\theta$ the characteristic excursion over ambient temperature, θ; and g the acceleration due to gravity (9.8 m s^{-2}). The flux of excess internal energy at x_c derives from that fraction of the heat generated sustaining no work on the engine, so

$$Q_{pe}C_v\Delta\theta = (1-\eta)Q_{fe}H_f\,, \tag{A.2}$$

where C_v is the specific heat capacity of the incompressible plume (718 J kg^{-1} K^{-1} under ISA sea-level conditions; the molar mass of exhausts in the stoichiometric combustion of a hydrocarbon of composition C_nH_{2n}, such as aviation fuel, is almost that of air); η the thermal efficiency of the engine; Q_{fe} the mass of fuel burnt per second; and H_f the heat liberated per unit mass of fuel (46.2 MJ kg^{-1} for aviation fuel prior to any condensation). Substituting for $Q_{pe}\cdot\Delta\theta$ in equation (A.1),

$$B_{Fe} = (1-\eta)g\,Q_{fe}H_f/(C_v\theta)\,\cdot \tag{A.3}$$

The compressible flow in the final working phase up to x_c may also be ascribed a characteristic velocity, Δu_c, satisfying

$$F_e = Q_{pe} \Delta u_c \text{,} \tag{A.4}$$

where F_e is the thrust developed by the engine. Efficiency η may thus be taken to satisfy the following,

$$\eta = Q_{pe} \Delta u_c^{\,2} \big/ (2 Q_{fe} H_f) = (Q_{pe}/Q_{fe})(F_e/Q_{pe})^2 \big/ (2H_f) \tag{A.5}$$

Thrust F_e scales with Q_{fe} to $O(1)$, according to standard emissions data, and in takeoff is of static equivalent in the range 75-100% of maximum rated output, depending on aircraft loading. Over this percentage range, η may be taken as constant to $O(1)$. From equation (A.3), it follows that B_{Fe} may be taken to scale with Q_{fe}, and thus with F_e. A time scale, τ, is obtained on dividing F_e by B_{Fe}, such that the buoyant force on emissions of characteristic age τ or younger is equal to F_e. Time τ thus parameterises the emission age at which the buoyancy becomes dynamically significant. A value for τ during takeoff may conveniently be established from characteristics of engines when static and at full burn, as shown in Figure A1. The trend of proportionality shown in Figure A1a is of gradient equal to the speed of sound under ISA sea-level conditions, c (341 m s^{-1}). The mass flux at x_c is that through the engine, to $O(1)$, and so Δu_c in equation (A.4) may, at top thrust, be taken as equal to c. The trend of proportionality shown in Figure A1b is of gradient, 310, so Q_{pe} at top thrust may be taken as a factor, 310, times Q_{fe}. It follows from equation (A.5) that η during takeoff is a factor, 155, times c^2/H_f, or 0.4. Similarly, it can be seen from equation (A.3) that τ during takeoff is a factor, 2.3, times c/g, or 80 s.

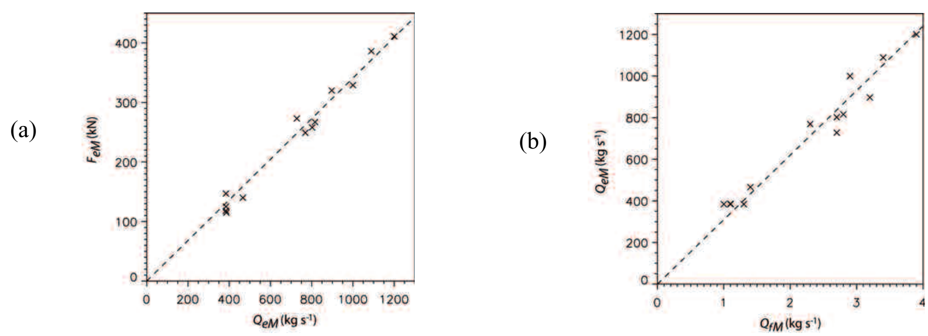

Figure A1. Relationships under ISA sea-level conditions between maximum static thrust of power plant, F_{eM}, airflow through the fan, Q_{eM}, and fuel flow, Q_{fM}, for the plant making up the Heathrow dataset (all of turbofan type): a) F_{eM} vs Q_{eM}, b) Q_{eM} vs Q_{fM} (data source: Jane's All the World's Aircraft, 2001-2). Dashed lines show fits described in the text.

REFERENCES

Högström, U., 1988: Non-dimensional wind and temperature profiles in the atmospheric surface layer: A re-evaluation. *Boundary-Layer Meteorol.*, **42**, 55-78.

Law, A. W.-K. and Herlina, 2002: An experimental study on turbulent circular wall jets. *J. Hydraul. Eng.*, **128**, 151-174.

Webb, E. K., 1982: Profile relationships in the superadiabatic surface layer. *Q. J. R. Meteorol. Soc.*, **108**, 661-688.